

GKM GRAPHS FOR ODD DIMENSIONAL MANIFOLDS WITH TORUS ACTIONS

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ABSTRACT. Let torus T act on a compact smooth manifold M , if the equivariant cohomology $H_T^*(M)$ is a free module of $H_T^*(pt)$, then according to the Chang-Skjelbred Lemma, $H_T^*(M)$ can be determined by the 1-skeleton M_1 consisting of fixed points and 1-dimensional orbits. Goresky, Kottwitz and MacPherson considered the case where M is an algebraic manifold and M_1 is 2-dimensional, and introduced a graphic description of equivariant cohomology. In this paper, we follow those ideas to consider the case where M is an odd-dimensional (possibly non-orientable) manifold and M_1 is 3-dimensional, and give a similar graphic description of equivariant cohomology.

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1. INTRODUCTION

Let torus T act on a compact smooth manifold M . The T -equivariant cohomology of M is defined using the Borel construction $H_T^*(M) = H^*((M \times ET)/T)$, where $ET = (S^\infty)^{\dim T}$ and the coefficient of cohomology will always be \mathbb{Q} throughout the paper. By this definition, if we denote \mathfrak{t}^* as the dual Lie algebra of T , then $H_T^*(pt) = H^*(ET/T) = H^*((\mathbb{C}P^\infty)^{\dim T}) = \mathbb{S}\mathfrak{t}^*$ is a polynomial ring in $\dim T$ variables. The trivial map $\iota : M \rightarrow pt$ induces a homomorphism $\iota^* : H_T^*(pt) \rightarrow H_T^*(M)$ and gives $H_T^*(M)$ a $H_T^*(pt)$ -module structure.

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For every point $p \in M$, its stabilizer is defined as $T_p = \{t \in T \mid t \cdot p = p\}$, and its orbit is $\mathcal{O}_p \cong T/T_p$. If we set the i -th skeleton $M_i = \{p \mid \dim \mathcal{O}_p \leq i\}$, then this gives an equivariant stratification $M_0 \subseteq M_1 \subseteq \dots \subseteq M_{\dim T} = M$ on M , where the 0-skeleton M_0 is exactly the fixed-point set M^T . If $H_T^*(M)$ is a free $H_T^*(pt)$ -module, also called equivariantly formal in [GKM98], Chang and Skjelbred [CS74] proved that $H_T^*(M)$ only depends on the fixed-point set M^T and the 1-skeleton M_1 .

Theorem (Chang-Skjelbred Lemma, [CS74]). *If torus T acts on M equivariantly formally, then the following short sequence is exact:*

$$0 \longrightarrow H_T^*(M) \longrightarrow H_T^*(M^T) \longrightarrow H_T^{*+1}(M_1, M^T)$$

This short exact sequence enables one to describe the equivariant cohomology $H_T^*(M)$ as a sub-ring of $H_T^*(M^T)$, subject to certain algebraic relations determined by the 1-skeleton M_1 . For example, Goresky, Kotwitz and MacPherson [GKM98] considered torus actions on algebraic varieties when the fixed-point set M^T is finite and the 1-skeleton M_1 is a union of spheres S^2 . They proved that the cohomology $H_T^*(M)$ can be described in terms of congruence relations on a regular graph determined by the 1-skeleton M_1 . Since then, various GKM-type theorems were proved, for instance, by Brion [Br97] on equivariant Chow groups, by Knutson&Rosu [KR03], Vezzosi&Vistoli [VV03] on equivariant K-theory, and by Guillemin&Holm [GH04] on Hamiltonian symplectic manifold with non-isolated fixed points. Recent generalization of GKM-type theorem is due to Goertsches, Nozawa&Töben [GNT12] on Cohen-Macaulay actions on K-contact manifolds, and Goertsches&Mare [GM14] on non-abelian actions.

In this paper, we will try to develop a graphic description of equivariant cohomology for manifolds (possibly non-orientable) in odd dimensional cases.

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2. TORUS ACTIONS AND EQUIVARIANT COHOMOLOGY

First we will recall some definitions and classical theorems regarding torus actions, equivariant cohomology (cf. [B72, AP93]) and stable almost complex manifolds (cf. [GGK02] Appendix D).

2.1. Torus actions and stable almost complex manifolds. Throughout the paper, a manifold M is always assumed to be compact, smooth and boundaryless. Let torus T act on a manifold M , we will denote M^T as the fixed-point set. For any point p in a connected component C of M^T , there is the **isotropy representation** of T on the tangent space $T_p M$, which splits into weighted spaces $T_p M = V_0 \oplus V_{[\alpha_1]} \oplus \dots \oplus V_{[\alpha_r]}$ where the non-zero distinct weights $[\alpha_1], \dots, [\alpha_r] \in \mathfrak{t}_{\mathbb{Z}}^*/\pm 1$ are determined only up to signs. Comparing with the tangent-normal splitting $T_p M = T_p C \oplus N_p C$, we get that $T_p C = V_0$ and $N_p C = V_{[\alpha_1]} \oplus \dots \oplus V_{[\alpha_r]}$. Since $N_p C = V_{[\alpha_1]} \oplus \dots \oplus V_{[\alpha_r]}$ is of even dimension, the dimensions of M and components of M^T will be of the same parity. If $\dim M$ is even, the smallest possible components of M^T could be isolated points. If $\dim M$ is odd, the smallest possible components of M^T could be isolated circles. Since T acts on the normal space $N_p C$ by rotation, this gives the normal space $N_p C$ an orientation. Moreover, if M is oriented, we see that any connected component C of M^T has an induced orientation.

A **stable almost complex structure** on M is a real vector bundle automorphism $J : TM \oplus \mathbb{R}^k \rightarrow TM \oplus \mathbb{R}^k$ for some positive integer k such that J fixes the base manifold M and $J^2 = -\text{Id}_{TM \oplus \mathbb{R}^k}$, i.e. $(TM \oplus \mathbb{R}^k, J)$ is a complex vector bundle over M . We will call such a triple $(M, TM \oplus \mathbb{R}^k, J)$ a **stable almost complex manifold**. On a given manifold M , two stable almost complex structures, $(TM \oplus \mathbb{R}^{k_1}, J_1)$ and $(TM \oplus \mathbb{R}^{k_2}, J_2)$, are **equivalent** if there is a complex vector bundle isomorphism $\varphi : (TM \oplus \mathbb{R}^{k_1} \oplus \mathbb{C}^{l_1}, J_1 \oplus I_1) \rightarrow (TM \oplus \mathbb{R}^{k_2} \oplus \mathbb{C}^{l_2}, J_2 \oplus I_2)$, where I_j is the canonical complex structure on \mathbb{C}^{l_j} .

Suppose now M has a stable almost complex structure $(TM \oplus \mathbb{R}^k, J)$, a T -action on M is said to be **compatible** with the stable almost complex structure, if J commutes with the T -action on $TM \oplus \mathbb{R}^k$, where \mathbb{R}^k is acted by T trivially. In this case, the isotropy representation $T \curvearrowright T_p M$ is a complex T -representation, hence we can determine the non-zero distinct isotropy weights $\alpha_1, \dots, \alpha_r \in \mathfrak{t}_{\mathbb{Z}}^*$ without ambiguity of signs. Since the normal space $N_p C = V_{\alpha_1} \oplus \dots \oplus V_{\alpha_r}$ is a complex T -subspace of $(T_p M \oplus \mathbb{R}^k = T_p C \oplus N_p C \oplus \mathbb{R}^k, J_p)$, we see that $T_p C \oplus \mathbb{R}^k$ is the T -subspace with 0-weight in $(T_p M \oplus \mathbb{R}^k, J_p)$, hence is also a complex subspace. Therefore, a connected component C of M^T has an induced stable almost complex structure $(TC \oplus \mathbb{R}^k, J_C)$, and hence has an induced orientation.

Remark 2.1.1. When M has a T -compatible stable almost complex structure, we get two orientations on a connected component C of M^T : either orient C with the complement orientation of the rotational orientation on the normal bundle of C , or orient C from its induced stable almost complex structure. These two orientations need not to be the same. For example, let S^1 act on S^2 by the standard rotation, then the fixed-point set consists of the north pole N and south pole S . If we orient the two poles with the complement orientation of the rotational orientation on their normal spaces, then we get opposite signs on N and S . However, if we think of S^2 as $\mathbb{C}P^1$ and give the two poles with the induced (trivial) complex orientation, then we get $+$ signs on both N and S (but the isotropy weights on the poles have opposite signs). Later on in the main Theorem 4.3.1, we will need to deal with the change of signs due to the two different induced orientations on fixed-point set.

For any subtorus K of T , we get two more actions automatically: the **sub-action** of K on M and the **residual action** of T/K on M^K .

2.2. Equivariant cohomology. Given a group action of G on M , the equivariant cohomology is defined using the Borel construction $H_G^*(M) \stackrel{\text{def}}{=} H^*((M \times EG)/G)$, which is a module of the ring $H_G^*(pt)$.

Comparing $H_G^*(M)$ with $H_G^*(M^G)$, a weak version of the Atiyah-Bott-Berline-Vergne (ABBV) localization formula says:

Theorem 2.2.1 (Atiyah-Bott-Berline-Vergne Localization Theorem). *The restriction map*

$$H_G^*(M) \longrightarrow H_G^*(M^G)$$

is a $H_G^(pt)$ -module isomorphism modulo $H_G^*(pt)$ -torsion.*

Inspired by this localization theorem, we can hope for more connections between the manifold M and its fixed-point set M^G , if $H_G^*(M)$ is actually $H_G^*(pt)$ -free.

Definition 2.2.1. An action of G on M is **equivariantly formal** if $H_G^*(M)$ is a free $H_G^*(pt)$ -module.

A direct consequence of the ABBV localization theorem 2.2.1 for equivariantly formal group action is:

Corollary 2.2.1 (Existence of fixed points). *If an action of G on M is equivariantly formal, then the fixed-point set M^G is non-empty.*

Proof. According to the ABBV localization theorem 2.2.1, the $H_G^*(M^G)$ will be of the same non-zero $H_G^*(pt)$ -rank as $H_G^*(M)$. Therefore, M^G is non-empty. \square

Using the techniques of spectral sequences, equivariant formality amounts to the degeneracy at E^2 level of the Leray-Serre sequence of the fibration $M \hookrightarrow (M \times EG)/G \rightarrow BG$.

In the case of torus action, there is a much more applicable criterion for equivariant formality, (cf. [AP93] Theorem 3.10.4).

Theorem 2.2.2 (Cohomology inequality and equivariant formality, [AP93]). *If a torus T acts on M , then $\sum \dim H^*(M^T) \leq \sum \dim H^*(M)$, where equality holds if and only if the action is equivariantly formal.*

A sufficient condition for equivariant formality is that

Corollary 2.2.2. *If a T -manifold M has a T -invariant Morse-Bott function f such that $\text{Crit}(f) = M^T$, then it is equivariantly formal.*

Proof. The cohomology $H^*(M)$ can be computed from Morse-Bott-Witten cochain complex generated on the critical submanifold $\text{Crit}(f)$. So $\sum \dim H^*(M^T) = \sum \dim H^*(\text{Crit}(f)) \geq \sum \dim H^*(M)$. Then the above theorem says this inequality is actually an equality and hence the T -manifold M is equivariantly formal. \square

Example 2.2.1. When M is equipped with a symplectic form, a Hamiltonian T -action and a moment map $\mu : M \rightarrow \mathfrak{t}^*$, then μ^ξ gives a Morse-Bott function for any generic $\xi \in \mathfrak{t}$ and has $\text{Crit}(\mu^\xi) = M^T$, therefore M is T -equivariantly formal.

Restricting to any subtorus K of T acting on M , we get

Proposition 2.2.1 (Inheritance of equivariant formality). *An action of torus T on M is equivariantly formal if and only if for any subtorus K of T , both the sub-action of K on M and the residual action of T/K on M^K are equivariantly formal.*

Proof. Notice that after choosing a subtorus K , the three actions of T on M , K on M and T/K on M^K give us the sequence of inequalities

$$\sum \dim H^*(M^T) \leq \sum \dim H^*(M^K) \leq \sum \dim H^*(M)$$

Thus, we see that the equality $\sum \dim H^*(M^T) = \sum \dim H^*(M)$ holds if and only if both of the two intermediate equalities $\sum \dim H^*(M^T) = \sum \dim H^*(M^K)$ and $\sum \dim H^*(M^K) = \sum \dim H^*(M)$ hold, which is just a restatement of the proposition. \square

Combining the Proposition 2.2.1 on inheritance of equivariant formality with the Corollary 2.2.1 on existence of fixed points, we get the inheritance of fixed points:

Corollary 2.2.3 (Inheritance of fixed points). *If an action of torus T on M is equivariantly formal, then for any subtorus K of T , every connected component of M^K has T -fixed points.*

Proof. By the inheritance of equivariant formality, the residual action of T/K on any connected component C of M^K is also equivariantly formal. Then by the existence of fixed points, $C^T = C^{T/K}$ is non-empty. \square

3. GKM THEORY IN EVEN DIMENSION

Goresky, Kottwitz and MacPherson[GKM98] originally considered their theory for algebraic manifolds. But the idea also works for general even dimensional oriented manifolds M^{2n} with torus action.

When a T -action on M is equivariantly formal, a simple application of the Atiyah-Bott-Berline-Vergne(ABBV) localization formula implies the non-emptiness of the fixed-point set M^T . Then the Chang-Skjelbred short exact sequence $0 \rightarrow H_T^*(M) \rightarrow H_T^*(M^T) \rightarrow H_T^{*+1}(M_1, M^T)$ says that one can describe the equivariant cohomology $H_T^*(M)$ by understanding

- (1) The generators from $H_T^*(M^T)$
- (2) The relations from $H_T^{*+1}(M_1, M^T)$

3.1. GKM condition in even dimension. To apply the Chang-Skjelbred Lemma, Goresky, Kottwitz and MacPherson[GKM98] considered the smallest possible 0-skeleton M^T and 1-skeleton M_1 .

Definition 3.1.1 (GKM condition in even dimension). An action of torus T on M^{2n} is **GKM** if

- (1) The fixed-point set M^T consists of isolated points.
- (2) At each fixed point $p \in M^T$, the non-zero weights $[\alpha_1], \dots, [\alpha_n] \in \mathfrak{t}_{\mathbb{Z}}^*/\pm 1$ of the isotropy T -representation $T \curvearrowright T_p M$ are pair-wise independent.

From the condition (1), we get $H_T^*(M^T) = \oplus_{p \in M^T} \mathbb{S}t^*$.

From the condition (2), at each fixed point p , we get pair-wise independent weights $[\alpha_1], \dots, [\alpha_n] \in \mathfrak{t}_{\mathbb{Z}}^*/\pm 1$ of the isotropy T -representation. If we denote T_{α_i} to be the subtorus of T with Lie sub-algebra $\mathfrak{t}_{\alpha_i} = \text{Ker } \alpha_i$, then the component C_{α_i} of $M^{T_{\alpha_i}}$ containing p will be of dimension 2 with the residual action of the circle T/T_{α_i} , i.e. a non-trivial S^1 -action on 2-dimensional surface with non-empty isolated fixed points.

3.2. The geometry and cohomology of 2d orientable S^1 -manifolds. According the classification of 2-dimensional orientable S^1 -manifolds with non-empty fixed points, there is a unique such manifold.

Lemma 3.2.1 (see [Au04]). *If S^1 acts on an oriented surface M with non-empty isolated fixed points, then M is the sphere S^2 , and the S^1 -action is a multiple of the standard spinning.*

Using equivariant Mayer-Vietoris sequence, we see that this action is equivariantly formal, with equivariant cohomology

$$H_{S^1}^*(S^2) = \{(f_N, f_S) \in \mathbb{Q}[u] \oplus \mathbb{Q}[u] \mid f_N(0) = f_S(0)\}$$

Transferring to the T -action on S^2 with subtorus T_α acting trivially and the residual circle T/T_α -action equivariantly formal, the equivariant cohomology is

$$H_T^*(S_{[\alpha]}^2) = H_{T/T_\alpha}^*(S_{[\alpha]}^2) \otimes H_{T_\alpha}^*(pt) = \{(f_N, f_S) \in \mathbb{S}t^* \oplus \mathbb{S}t^* \mid f_N \equiv f_S \pmod{\alpha}\}$$

gives relations of elements of $H_T^*(M)$ expressed in terms of $H_T^*(M^T)$.

3.3. GKM graph and GKM theorem in even dimension. Thanks to the simple Fact 3.2.1, now it is clear that, under GKM condition 3.1.1, the 0-skeleton is the fixed-point set M^T , and the 1-skeleton is the union of all the spheres $S_{[\alpha]}^2$ joined together at the fixed points. Thinking of the generators M^T as vertices, the relations $M^{T_{[\alpha]}}$ as edges, Goresky, Kottwitz and MacPherson associate a regular graph to each GKM torus action:

Definition 3.3.1 (GKM graph in even dimension). The **GKM graph** of a GKM action of torus T on M^{2n} consists of

Vertices: The fixed-point set M^T

Edges & Weights: An edge joins two fixed points if a sphere $S_{[\alpha]}^2 \subset M_1$ joins these two fixed points as North and South poles, and will be weighted $[\alpha]$.

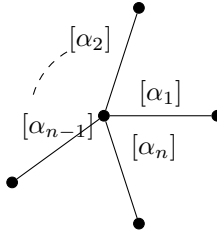


FIGURE 1. Each vertex has exactly n neighbours

Remark 3.3.1. If M^{2n} has a T -invariant stable almost complex structure, then the isotropy weights $\alpha_1, \dots, \alpha_n \in \mathfrak{t}_{\mathbb{Z}}^*$ are determined with signs, and its GKM graph can be made into a directed graph. Moreover, as explained by Guillemin and Zara [GZ01], there is a set of congruence relations between the bouquets of isotropy weights for each edge, and they call it the **connection** of the GKM graph.

Remark 3.3.2. If a GKM T -manifold M^{2n} is symplectic with a moment map $\mu : M \rightarrow \mathfrak{t}^*$, then the GKM graph is exactly the image $\mu(M_1)$.

Now, we can describe the equivariant cohomology $H_T^*(M)$ in a graph-theoretic way.

Theorem 3.3.1 (GKM theorem in even dimension, [GKM98]). *If an action of torus T on an oriented manifold M^{2n} is equivariantly formal and GKM, we can construct its GKM graph Γ , with vertex set $V = M^T$ and weighted edge set E , then the equivariant cohomology is*

$$H_T^*(M) = \{f : V \rightarrow \mathbb{S}t^* \mid f_x \equiv f_y \pmod{\alpha} \text{ for each edge } \overline{xy} \text{ with weight } [\alpha] \text{ in } E\}$$

Proof. Combining Chang-Skjelbred Lemma and the equivariant cohomology of S^2 , we get the GKM theorem. \square

Remark 3.3.3. Suppose a GKM manifold M has k connected components M_1, \dots, M_k , then $\dim H_T^0(M) = \dim H^0(M) = k$. Suppose its GKM graph has l connected components $\Gamma_1, \dots, \Gamma_l$. Note the assignment of polynomials on vertices from the same Γ_i with the same constant rational number gives all the elements in the graphic description of $H_T^0(M)$. Thus we have $\dim H_T^0(M) = l$. Therefore, $k = l$, the connectedness of M is exactly the connectedness of its GKM graph Γ . Especially, if M is connected then the graph Γ is also connected.

Example 3.3.1. Toric manifolds are GKM manifolds.

Interesting examples of Hamiltonian, non-toric GKM manifolds can be found in Guillemin and Zara's work [GZ01]. But there could also be non-symplectic GKM manifolds.

Example 3.3.2. For the sphere S^{2n} , we use the coordinates (x, z_1, \dots, z_n) where x is a real variable, z_i 's are complex variables. Let T^n act on S^{2n} by $(e^{i\theta_1}, \dots, e^{i\theta_n}) \cdot (x, z_1, \dots, z_n) = (x, e^{i\theta_1} z_1, \dots, e^{i\theta_n} z_n)$ with fixed-point set $(S^{2n})^{T^n} = \{(\pm 1, 0, \dots, 0)\}$. Since $\dim H^*((S^{2n})^{T^n}) = 2 = \dim H^*(S^{2n})$, the T^n action on S^{2n} is equivariantly formal by the formality criterion Theorem 2.2.2. Let $\alpha_1, \dots, \alpha_n$ be the standard integral basis of $\mathbb{t}_{\mathbb{Z}}^* = \mathbb{Z}^n$, then each fixed point has the unsigned isotropy weights $[\alpha_1], \dots, [\alpha_n]$. This means the action is GKM and the GKM graph consists of two vertices with n edges weighted $[\alpha_1], \dots, [\alpha_n]$ joining them. The equivariant cohomology is then $H_{T^n}^*(S^{2n}) = \{(f, g) \in \mathbb{S}t^* \oplus \mathbb{S}t^* \mid f \equiv g \pmod{\prod_{i=1}^n \alpha_i}\}$.

Remark 3.3.4. S^{2n} , $n \geq 2$ is not symplectic because $H^2(S^{2n}) = 0$, $n \geq 2$. Moreover, when n is large, S^{2n} does not even have almost complex structure, proved by Wu [Wu52] for S^{4n} , $n \geq 1$ and more generally by Borel and Serre [BS53] for S^{2n} , $n \geq 4$. This is why it is not necessary to require an even dimensional GKM manifold to have almost complex structure.

Note that the classical oriented GKM theorem comes directly from the Chang-Skjelbred lemma which has no assumption on the orientability, we will apply the Chang-Skjelbred lemma under the GKM condition 3.1.1 for manifolds not necessarily orientable.

3.4. The geometry and cohomology of 2d S^1 -manifolds. When M is not oriented, the 1-skeleton M_1 does not have to be orientable, but M_1 is still 2-dimensional due to the GKM condition 3.1.1 and again has a nice description of its geometry and cohomology.

Lemma 3.4.1 (see [Au04]). *If S^1 acts effectively on a surface M with non-empty isolated fixed points, then M is*

- S^2 with two fixed points
- $\mathbb{R}P^2$ with one fixed point, and an exceptional orbit $S^1/(\mathbb{Z}/2\mathbb{Z})$

where $\mathbb{R}P^2$ as the $\mathbb{Z}/2\mathbb{Z}$ quotient of S^2 , has the induced S^1 -action from S^2 .

Using equivariant Mayer-Vietoris sequence, we see that the S^1 -action on $\mathbb{R}P^2$ is also equivariantly formal, with equivariant cohomology

$$H_{S^1}^*(\mathbb{R}P^2) = \mathbb{Q}[u]$$

Transferring to the T -action on $\mathbb{R}P^2$ with subtorus T_β acting trivially and the residual circle T/T_β -action equivariantly formal, the equivariant cohomology is

$$H_T^*(\mathbb{R}P_{[\beta]}^2) = \mathbb{S}t^*$$

that gives relations of elements of $H_T^*(M)$ expressed in terms of $H_T^*(M^T)$.

3.5. The generalized GKM graph and GKM theorem in even dimension. In the 1-skeleton M_1 , the residual S^1 -actions on each $\mathbb{R}P^2$ still have two non-principal orbits but with different types: a fixed point and a $S^1/(\mathbb{Z}/2\mathbb{Z})$ orbit. Using these observation, we will construct a slightly different graph to represent the relation among M^T and M_1 .

Definition 3.5.1 (Generalized GKM graph in even dimension). The **GKM graph** of a GKM action of torus T on M^{2n} consists of

Vertices: There are two types of vertices

•: for each fixed point in M^T

Empty dot: for each exceptional orbit in a $\mathbb{R}P^2_{[\beta]} \in M_1$

Edges & Weights: A solid edge with weight $[\alpha]$ for each $S^1_{[\alpha]}$ joining two •'s representing its two fixed points, and a dotted edge with weight $[\beta]$ for each $\mathbb{R}P^2_{[\beta]}$ joining a • to an empty vertex.

Remark 3.5.1. By the GKM condition 3.1.1, a fixed point has exactly n pair-wise independent weights. Thus each •, representing a fixed point, is joined by exactly n edges to •'s or empty vertices. Note that each empty vertex is an exceptional orbit in a projective space $\mathbb{R}P^2_{[\alpha]} \subseteq M^{T_\alpha}$, hence belongs to a unique such $\mathbb{R}P^2$ and will have exactly one edge joining it to the fixed point of the $\mathbb{R}P^2_{[\alpha]}$. Because of this one-to-one correspondence between a $\mathbb{R}P^2_{[\alpha]} \in M_1$ and its exceptional orbit, any empty vertex is not shared by two different $\mathbb{R}P^2_{[\alpha_1]}, \mathbb{R}P^2_{[\alpha_2]} \in M_1$.

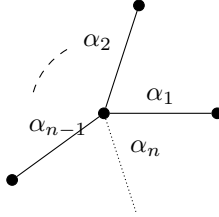


FIGURE 2. Each • has exactly n edges

The generalization of GKM theorem to even-dimensional non-orientable manifolds is due to Goertsches and Mare [GM14].

Theorem 3.5.1 (Generalized GKM theorem in even dimension, [GM14]). *If the action of torus T on a (possibly non-orientable) manifold M^{2n} is equivariantly formal and GKM, then we can construct its GKM graph Γ , with vertex set $V = M^T$ and weighted edge set E , moreover the equivariant cohomology has a graphic description*

$$H_T^*(M) = \{f : V \rightarrow \mathbb{S}t^* \mid f_p \equiv f_q \pmod{\alpha} \text{ for each solid edge } \overline{pq} \text{ with weight } \alpha \text{ in } E\}$$

Proof. Combining Chang-Skjelbred Lemma and the equivariant cohomology of S^2 and $\mathbb{R}P^2$, we get the GKM theorem. \square

Remark 3.5.2. The $\mathbb{R}P^2$'s in the 1-skeleton M_1 don't contribute to the congruence relations. We can erase all the dotted edges in the GKM graph, and call the remaining graph as the **effective** GKM graph, though the graph now is not regular, i.e. the vertices no longer have the same number of edges.

Remark 3.5.3. Note that in this paper we are working in \mathbb{Q} coefficients. However, if we want to get a GKM-type theorem for much subtler coefficients like \mathbb{Z} , the $\mathbb{R}P^2$'s in the 1-skeleton M_1 and their corresponding dotted edges in the GKM graph are as crucial as the S^2 's and their corresponding solid edges.

Example 3.5.1. $\mathbb{R}P^{2n}$ as the quotient of S^{2n} by the $\mathbb{Z}/2\mathbb{Z}$ action $e^{\pi i} \cdot (x, z_1, \dots, z_n) = (-x, -z_1, \dots, -z_n)$ also inherits a T^n -action from that on S^{2n} , discussed in previous section. The fixed-point set is $(\mathbb{R}P^{2n})^{T^n} = \{(\pm 1, 0, \dots, 0)\}/(\mathbb{Z}/2\mathbb{Z})$, a single point. Since $\dim H^*((\mathbb{R}P^{2n})^{T^n}) = 1 = \dim H^*(\mathbb{R}P^{2n})$, the T^n action on $\mathbb{R}P^{2n}$ is equivariantly formal by the formality criterion Theorem 2.2.2 with the unsigned isotropy weights $[\alpha_1], \dots, [\alpha_n]$ at the only fixed point. This means the action is GKM and the GKM graph consists of a single vertex with n dotted edges weighted $[\alpha_1], \dots, [\alpha_n]$, and the effective GKM graph is a single vertex without edges. The equivariant cohomology is then $H_{T^n}^*(\mathbb{R}P^{2n}) = \mathbb{S}t^*$.

4. GKM THEORY IN ODD DIMENSION

With the even dimensional GKM theory well established, it is natural to ask whether there is a parallel odd dimensional GKM theory. Goertsches, Nozawa and Töben [GNT12] developed a GKM theory for a certain class of Cohen-Macaulay torus actions on manifolds in arbitrary dimension, including an application to certain K-contact manifolds. In this paper, we will first introduce an odd dimensional GKM theory for torus actions on stable almost complex manifolds, then extend it for general manifolds (possibly non-orientable).

4.1. GKM condition in odd dimension. As we have seen in the even dimensional case, the essence of GKM theory is to find an ideal condition for the application of the Change-Skjelbred Lemma. Here is the odd dimensional version of the GKM condition:

Definition 4.1.1 (GKM condition in odd dimension). An action of torus T on stable almost complex manifold $(M^{2n+1}, TM \oplus \mathbb{R}^k, J)$ is **GKM** if

- (1) The fixed-point set M^T consists of isolated circles.
- (2) Along each fixed circle $\gamma \subset M^T$, the non-zero weights $\alpha_1, \dots, \alpha_n \in \mathfrak{t}_{\mathbb{Z}}^*$ of the isotropy T -representation $T \curvearrowright T_{\gamma}M$ are pair-wise independent.

From the condition (1), the fixed-point set M^T consists of circles γ 's. As we have observed in the preparation Subsection 2.1, for a torus action on stable almost complex manifold, the fixed-point set M^T is canonically oriented. Then we can use the orientation form θ_{γ} to be the generator of $H^1(\gamma)$, and write

$$H_T^*(M^T) = \oplus_{\gamma \subset M^T} (H_T^*(pt) \otimes H^*(S_{\gamma}^1)) = \oplus_{\gamma \subset M^T} (\mathbb{S}t^* \oplus \mathbb{S}t^* \theta_{\gamma})$$

From the condition (2), similar to the even dimensional case, along each fixed circle $\gamma \subset M^T$, we get pair-wise independent weights $\alpha_1, \dots, \alpha_n \in \mathfrak{t}_{\mathbb{Z}}^*$ of the isotropy T -representation. If we denote T_{α_i} to be the subtorus of T with Lie sub-algebra $\mathfrak{t}_{\alpha_i} = \text{Ker } \alpha_i$, then the component C_{α_i} of $M^{T_{\alpha_i}}$ containing γ will be of dimension 3 with the residual action of the circle T/T_{α_i} , i.e. a non-trivial S^1 -action on 3-dimensional manifold with non-empty isolated fixed points.

4.2. The geometry and cohomology of 3d orientable S^1 -manifolds. 3-dimensional S^1 -manifolds without fixed points were classified by Seifert, hence are named as Seifert manifolds. The case of 3-dimensional S^1 -manifolds with or without fixed points, also called as generalized Seifert manifolds, were classified by Orlik and Raymond.

Briefly speaking, the equivariant diffeomorphism type of a 3-dimensional S^1 -manifold M^3 is determined by the orbifold type of its quotient space M/S^1 , the numeric data of the Seifert fibres over orbifold points of M/S^1 , and the orbifold Euler number of the “fibration” $M \rightarrow M/S^1$.

For the interest of this paper, we will need a classification result of 3-dimensional closed S^1 -manifolds with non-empty fixed points.

Let's denote the genus of the orbifold surface M/S^1 as g , the positive number of connected components in the non-empty M^{S^1} as f , and a pair of Seifert invariants as (μ_i, ν_i) .

Theorem 4.2.1 (Orlik-Raymond classification of closed S^1 -manifolds, [Ra68, OR68]). *There is an one-to-one correspondence from the equivariant diffeomorphism types of closed oriented 3d S^1 -manifolds with non-empty fixed-point sets to the orbit invariants:*

$$\left\{ \text{Effective } S^1 \curvearrowright M^3 \mid M^{S^1} \neq \emptyset \right\} / \text{Equiv diffeoms} \xrightarrow{\cong} \left\{ (g, f > 0, (\mu_1, \nu_1), \dots, (\mu_r, \nu_r)) \mid \mu_i, \nu_i \text{ coprime}, 0 < \nu_i < \mu_i \right\}$$

The proof of this theorem is by equivariant cutting and pasting, and furthermore inspires one to compute its equivariant cohomology using Mayer-Vietoris sequences.

For a closed oriented S^1 -manifold $M = \{g, f, (\mu_1, \nu_1), \dots, (\mu_r, \nu_r)\}$ with non-empty fixed-point set, its equivariant cohomology comes from two parts: the fixed-point set M^{S^1} and the fixed-point-free set $M \setminus M^{S^1}$.

The fixed-point set M^{S^1} consists of circles γ_i 's. Let's use δ_i for the generators of $H^0(\gamma_i)$. Moreover, passing to the orbit space M/S^1 , the fixed circles contribute as boundaries, i.e. $\gamma_i \subset \partial(M/S^1)$. We then choose the canonical boundary orientation forms θ_i as the generators of their $H^1(\gamma_i)$.

The fixed-point-free set $M \setminus M^{S^1}$ has a quotient space $(M \setminus M^{S^1})/S^1$, the orbifold surface of genus g with f boundary circles and r orbifold points. Since we only consider cohomology in \mathbb{Q} coefficients, the orbifold points won't actually appear in the cohomology calculations. Let's use δ_0 and $\alpha_j, \beta_j, 0 \leq j \leq g$ for the generators of H^0 and H^1 of the surface with genus g .

Theorem 4.2.2 (Equivariantly cohomology of 3d S^1 -manifold, [He]). *For a closed oriented 3d S^1 -manifold $M = \{g, f, (\mu_1, \nu_1), \dots, (\mu_r, \nu_r)\}$ with non-empty fixed-point set, an element of its equivariant cohomology $H_{S^1}^*(M)$ can be written as*

$$\left(D\delta_0 + \sum_j (A_j\alpha_j + B_j\beta_j) + \sum_i C_i\theta_i; \sum_i (P_i(u)\delta_i + Q_i(u)\theta_i) \right)$$

in $H^*(M/S^1) \oplus \oplus_i (\mathbb{Q}[u] \otimes H^*(\gamma_i))$, where $A_j, B_j, C_i, D \in \mathbb{Q}$ and $P_i(u), Q_i(u) \in \mathbb{Q}[u]$, under the relations

- (1) $\sum_j (A_j + B_j) + \sum_i C_i = 0$
- (2) $P_1(0) = P_2(0) = \dots = P_f(0) = D$
- (3) $Q_i(0) = C_i$ for each i

Especially, when the S^1 -action on M^3 is equivariantly formal, the cohomology $H_{S^1}^*(M)$ will be a subring of the cohomology $H_{S^1}^*(M^{S^1})$ with a more succinct description.

Theorem 4.2.3 (Equivariant formal 3d orientable S^1 -manifold, [He]). *A closed oriented 3d S^1 -manifold $M = \{g, f > 0, (\mu_1, \nu_1), \dots, (\mu_r, \nu_r)\}$, with non-empty fixed-point sets, is equivariantly formal if and only if $g = 0$.*

Moreover, in this case, the equivariant cohomology has a much nicer expression of the equivariant cohomology $H_{S^1}^(M)$:*

$$\sum_{i=1}^f (P_i(u)\delta_i + Q_i(u)\theta_i) \in \oplus_i (\mathbb{Q}[u] \otimes H^*(\gamma_i))$$

under the relations:

$$P_1(0) = P_2(0) = \dots = P_f(0) \text{ and } \sum_{i=1}^f Q_i(0) = 0$$

Transferring to a T -action on M^3 with subtorus T_α acting trivially and the residual circle T/T_α -action equivariantly formal, the equivariant cohomology $H_T^*(M_\alpha^3)$ can be given as:

$$\sum_{i=0}^f (P_i\delta_i + Q_i\theta_i) \in \oplus_i (\text{St}^* \otimes H^*(\gamma_i))$$

where $P_i, Q_i \in \mathbb{St}^*$ are polynomials, under the relations:

$$(\dagger) \quad P_1 \equiv P_2 \equiv \cdots \equiv P_f \text{ and } \sum_{i=1}^f Q_i \equiv 0 \pmod{\alpha}$$

4.3. GKM graph and GKM theorem in odd dimension. Similar to the original even dimensional GKM theory, we will construct GKM graphs for odd dimensional GKM manifolds and give a graph-theoretic computation of their equivariant cohomology.

In the even dimensional case, the unique 2d S^1 -manifold with fixed points is the sphere S^2 with exactly 2 fixed points. Each of such sphere gives rise to an edge connecting the 2 fixed points in GKM graphs. It is this simple fact that makes the GKM graphs in even dimensions regular, i.e. all the vertices have the same number of connecting edges.

However, in odd dimension, as we have seen in the previous discussion on 3d S^1 -manifold with fixed points, there could be any positive number of fixed components, in contrast to the exactly 2 fixed points of S^2 . Due to this difference, we need to modify the original construction of GKM graphs a bit.

Definition 4.3.1 (GKM graph in odd dimension). The **GKM graph** for a GKM action of torus T on stable almost complex manifold $(M^{2n+1}, TM \oplus \mathbb{R}^k, J)$ consists of

Vertices: There will be two types of vertices.

◦ for each fixed circle $\gamma \subset M^T$.

◻ for each 3d connected component C_α^3 in M^{T_α} of some subtorus T_α of codimension 1.

Edges & Weights: An edge joins a (\square, C) to a (\circ, γ) , if the 3d manifold C contains the fixed circle γ and hence is a connected component of M^{T_α} for an isotropy weight α of γ . The edge is then weighted with α . There are no edges directly joining ◦ to ◦, nor ◻ to ◻.

Remark 4.3.1. By the GKM condition 4.1.1, a fixed circle has exactly n pair-wise independent weights. Thus each ◦, representing a fixed circle, is joined by exactly n edges to n ◻'s. Notice that C as a connected component of M^{T_α} , can contain any positive number of fixed circles, and is also a connected component of M^{T_α} . Thus each ◻, representing a 3d component, can be joined by any positive number of edges to ◦'s, with weight either α or $-\alpha$ (both can appear).

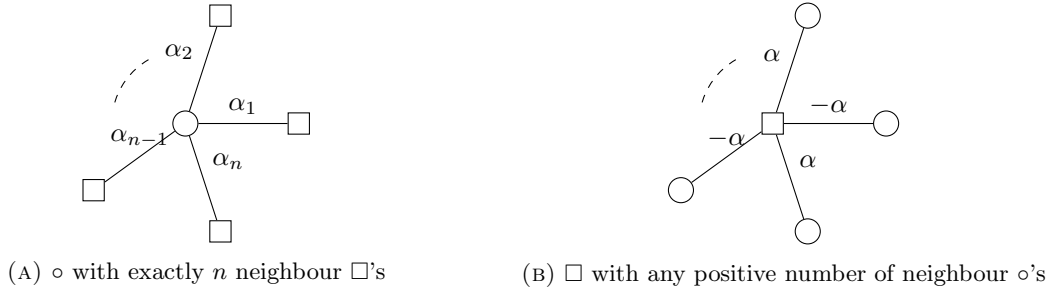


FIGURE 3. Neighbourhoods of the two types of vertices

Let's describe a GKM-type theorem for the equivariant cohomology $H_T^*(M^{2n+1})$ in a graph-theoretic way

Theorem 4.3.1. *If an action of torus T on stable almost complex manifold $(M^{2n+1}, TM \oplus \mathbb{R}^k, J)$ is equivariantly formal and GKM, we can construct its GKM graph as Γ , with two types of vertex sets V_\circ and V_\square and weighted edge set E , then an element of the equivariant cohomology $H_T^*(M)$ can be written as:*

$$(P, Q\theta) : V_\circ \longrightarrow \mathbb{St}^* \oplus \mathbb{St}^*\theta$$

where θ is the generator of $H^1(S^1)$, under the relations that for each \square representing a 3d component C of some M^{T^α} and the neighbour \circ 's representing the fixed circles $\gamma_1, \dots, \gamma_k$ on this component,

$$(\ddagger) \quad P_{\gamma_1} \equiv P_{\gamma_2} \equiv \dots \equiv P_{\gamma_k} \text{ and } \sum_{i=1}^k \pm Q_{\gamma_i} \equiv 0 \pmod{\alpha}$$

where the signs are specified by the weights of edges connecting the (\square, C) to the (\circ, γ_i) .

Proof. The GKM condition implies that the 0-skeleton M^T is a set of isolated circles, and that the 1-skeleton M_1 is a union of 3d manifolds with residual circle actions and non-empty fixed-point sets. The equivariant formality enables one to apply the Chang-Skjelbred Lemma.

The generators of $H_T^*(M)$ are given by the $H_T^*(M^T) = \oplus_{\gamma \subset M^T} (\text{St}^* \otimes H^*(\gamma))$. In other words, to each fixed circle γ which is represented as a $\circ \in V_\circ$, we associate a pair of polynomials $(P_\gamma, Q_\gamma \theta_\gamma) \in \text{St}^* \otimes H^*(\gamma)$.

By the Proposition 2.2.1 on inheritance of equivariant formality, every 3d T/T_α -component C , represented by a $\square \in V_\square$, is also equivariantly formal. Then we can use the Classification Theorem 4.2.3 of equivariantly formal S^1 -actions on closed 3d manifolds, and the relations \ddagger therein.

The only modifications are the signs in $\sum_{i=1}^k \pm Q_{\gamma_i}$. Notice that in the Theorem 4.2.3, the orientation forms θ_γ are chosen to be compatible with the orientation of the component $C \subset M^{T^\alpha}$, such that the isotropy weight of γ is exactly 1 under the residual $S^1 = T/T_\alpha$ -action, or equivalently with weight α under the T -action. However, if we orient the γ as a fixed component in M^T , then as in Remark 4.3.1, the involved weights could be α or $-\alpha$, whose signs are the adjustment for the relation \ddagger . Since the choice of orientation form θ_γ for $\gamma \subset M^T$ with induced stable almost complex structure is canonical, we can drop the subscript and simply write θ universally as the orientation form for every γ . \square

Remark 4.3.2. To describe the additive, multiplicative and St^* -module structure, it is convenient to write an element $(P, Q\theta)$ as $(P_\gamma + Q_\gamma \theta)_{\gamma \subset M^T}$. Note $\theta^2 = 0$, then $(P_\gamma + Q_\gamma \theta)_{\gamma \subset M^T} + (\bar{P}_\gamma + \bar{Q}_\gamma \theta)_{\gamma \subset M^T} = ([P_\gamma + \bar{P}_\gamma] + [Q_\gamma + \bar{Q}_\gamma]\theta)_{\gamma \subset M^T}$, and $(P_\gamma + Q_\gamma \theta)_{\gamma \subset M^T} \cdot (\bar{P}_\gamma + \bar{Q}_\gamma \theta)_{\gamma \subset M^T} = ([P_\gamma \bar{P}_\gamma] + [P_\gamma \bar{Q}_\gamma + \bar{P}_\gamma Q_\gamma]\theta)_{\gamma \subset M^T}$. For any polynomial $R \in \text{St}^*$, we have $R \cdot (P_\gamma + Q_\gamma \theta)_{\gamma \subset M^T} = (RP_\gamma + RQ_\gamma \theta)_{\gamma \subset M^T}$.

4.4. Further generalization. We can further weaken the assumptions in the above Theorem 4.3.1:

- (1) Notice that the core argument of Theorem 4.3.1 is only based on the Chang-Skjelbred Lemma and the Theorem 4.2.3 of 3d equivariantly formal orientable S^1 -manifold, hence also works for oriented manifolds. The main purpose of requiring M being stable almost complex is to help determine the signs of the isotropy weights $\alpha \in \mathbb{Z}_T^*$ and the signs of the $\sum_{i=1}^k \pm Q_{\gamma_i}$ in relation \ddagger . If M is oriented without specifying stable almost complex structure, then the isotropy weights are only determined as $[\alpha_1], \dots, [\alpha_k] \in \mathbb{Z}_T^*/\pm 1$. However, we can still obtain the relation \ddagger , but need to determine the signs of $\sum_{i=1}^k \pm Q_{\gamma_i}$ in an ad hoc way by choosing in advance the orientations of M^T and M^{T^α} then comparing them.
- (2) Not only can we drop the requirement of M being stable almost complex, we can also apply Chang-Skjelbred Lemma to non-orientable manifolds in odd dimensional case to obtain a graphic description of equivariant cohomology.

We need a non-orientable version of Theorem 4.2.3.

Theorem 4.4.1 (Equivariantly formal 3d non-orientable S^1 -manifold, [He]). *A closed non-orientable 3d S^1 -manifold $M = \{g, \epsilon, f > 0, s, (\mu_1, \nu_1), \dots, (\mu_r, \nu_r)\}$, with non-empty fixed-point sets, is equivariantly formal if and only if*

$$\begin{cases} g = 0, s = 1 & \text{if } \epsilon = o \\ g = 1, s = 0 & \text{if } \epsilon = n \end{cases}$$

Moreover, in this case, the equivariant cohomology $H_{S^1}^*(M)$ is:

$$\sum_{i=0}^f (P_i(u)\delta_i + Q_i(u)\theta_i) \in \mathbb{Q}[u] \otimes H^*(F)$$

under the relations:

$$P_1(0) = P_2(0) = \cdots = P_f(0)$$

Remark 4.4.1. In the above theorem, the ϵ is the orientability of the orbits space M/S^1 , the s is the number of special exceptional orbits. For more details, see [He].

Remark 4.4.2. Although there are two types of equivariantly formal 3d non-orientable S^1 -manifolds, their equivariant cohomologies have the unified expression. Comparing with the orientable case in Theorem 4.2.3, the non-orientable case does not have constraints on the Q_i polynomials.

Transferring to a T -action on a non-orientable M^3 with subtorus T_α acting trivially and the residual circle T/T_α -action equivariantly formal, the equivariant cohomology $H_T^*(M_\alpha^3)$ can be given as:

$$\sum_{i=0}^f (P_i\delta_i + Q_i\theta_i) \in \oplus_i (\mathbb{S}t^* \otimes H^*(\gamma_i))$$

where $P_i, Q_i \in \mathbb{S}t^*$ are polynomials, under the relations:

$$P_1 \equiv P_2 \equiv \cdots \equiv P_f \pmod{\alpha}$$

For an odd dimensional (possibly non-orientable) T -manifold M , the isotropy weights $[\alpha_1], \dots, [\alpha_r] \in \mathfrak{t}_{\mathbb{Z}}^*/\pm 1$ at a component of M^T are only determined up to signs. The GKM condition 4.1.1 (2) in this case should be stated for $[\alpha_1], \dots, [\alpha_r] \in \mathfrak{t}_{\mathbb{Z}}^*/\pm 1$ to be pair-wise independent. Correspondingly, the weights of the edges in GKM graph are unsigned.

To get a similar GKM-type theorem for equivariant cohomology $H_T^*(M)$, as we observed at the beginning of this subsection, we need to fix in advance an orientation θ_i for each fixed circle $\gamma_i \subseteq M^T$, and also fix an orientation for each orientable $M^{T_\alpha} \subseteq M_1$.

Theorem 4.4.2. *If an action of torus T on (possibly non-orientable) manifold M^{2n+1} is equivariantly formal and GKM, then we can construct its GKM graph as Γ , with two types of vertex sets V_\circ and V_\square and weighted edge set E , then an element of the equivariant cohomology $H_T^*(M)$ can be written as:*

$$(P, Q\theta) : V_\circ \longrightarrow \mathbb{S}t^* \oplus \mathbb{S}t^*\theta$$

where θ is the generator of $H^1(S^1)$, under the relations that for each \square representing a 3d component C of some M^{T_α} and the neighbour \circ 's representing the fixed circles $\gamma_1, \dots, \gamma_k$ on this component,

- if C is non-orientable,

$$P_{\gamma_1} \equiv P_{\gamma_2} \equiv \cdots \equiv P_{\gamma_k} \pmod{\alpha}$$

- if C is orientable,

$$P_{\gamma_1} \equiv P_{\gamma_2} \equiv \cdots \equiv P_{\gamma_k} \text{ and } \sum_{i=1}^k \pm Q_{\gamma_i} \equiv 0 \pmod{\alpha}$$

where the sign for each Q_{γ_i} is specified by comparing the predetermined orientation θ_i with the induced orientation of C on γ_i .

Proof. Same as Theorem 4.3.1. □

Remark 4.4.3. If we reverse the predetermined orientation on a $\gamma_i \subseteq M^T$, then we just replace Q_{γ_i} by $-Q_{\gamma_i}$. If we reverse the predetermined orientation on an orientable component C of $M^{T_\alpha} \subseteq M_1$, then we just replace $\sum_{i=1}^k \pm Q_{\gamma_i}$ by $\sum_{i=1}^k \mp Q_{\gamma_i}$. Therefore, different choices of predetermined orientations give isomorphic equivariant cohomology.

4.5. Examples. Next, we will give some examples of odd-dimensional GKM T -manifolds and apply the Theorem 4.3.1 to describe equivariant cohomology with help of graphs.

Example 4.5.1. All the 3d S^1 -equivariantly formal manifolds, that we used in Theorem 4.2.3, are the building blocks of the odd dimensional GKM theory. For any S^1 -equivariantly formal orientable manifold $M^3 = \{g = 0, \epsilon = o, f > 0, s = 0, (\mu_1, \nu_1), \dots, (\mu_r, \nu_r)\}$, the GKM graph consists of one \square -vertex, representing the manifold M , with $f > 0$ edges of weight 1 joining to f \circ -vertices, representing the f fixed circles. The equivariant cohomology is $H_{S^1}^*(M) = \{(P_1, Q_1\theta; \dots; P_f, Q_f\theta) \in (\mathbb{Q}[u] \oplus \mathbb{Q}[u]\theta)^{\oplus f} \mid P_1(0) = \dots = P_f(0), \sum_{i=1}^f Q_i(0) = 0\}$. For any S^1 -equivariantly formal non-orientable manifold $M^3 = \{g = 0, \epsilon = o, f > 0, s = 1, (\mu_1, \nu_1), \dots, (\mu_r, \nu_r)\}$ or $\{g = 1, \epsilon = n, f > 0, s = 0, (\mu_1, \nu_1), \dots, (\mu_r, \nu_r)\}$, the GKM graph is the same as the case of oriented case, but the equivariant cohomology is $H_{S^1}^*(M) = \{(P_1, Q_1\theta; \dots; P_f, Q_f\theta) \in (\mathbb{Q}[u] \oplus \mathbb{Q}[u]\theta)^{\oplus f} \mid P_1(0) = \dots = P_f(0)\}$.

Example 4.5.2. For the sphere S^{2n+1} , we use the coordinates (z_0, z_1, \dots, z_n) where z_i 's are complex variables. Let T^n act on S^{2n+1} by $(e^{i\theta_1}, \dots, e^{i\theta_n}) \cdot (z_0, z_1, \dots, z_n) = (z_0, e^{i\theta_1}z_1, \dots, e^{i\theta_n}z_n)$ with fixed-point set $(S^{2n+1})^{T^n} = \{(z_1, 0, \dots, 0) \mid |z_1| = 1\} \cong S^1$. Since $\sum \dim H^*((S^{2n+1})^{T^n}) = 2 = \sum \dim H^*(S^{2n+1})$, the T^n action on S^{2n+1} is equivariantly formal by the formality criterion Theorem 2.2.2. Let $\alpha_1, \dots, \alpha_n$ be the standard integral basis of $\mathbb{t}_{\mathbb{Z}}^* = \mathbb{Z}^n$, then the unique fixed circle has the unsigned isotropy weights $[\alpha_1], \dots, [\alpha_n]$. This means the action is GKM and the GKM graph consists of one \circ -vertex with n edges weighted $[\alpha_1], \dots, [\alpha_n]$ joining to n \square -vertices. The equivariant cohomology is then $H_{T^n}^*(S^{2n+1}) = \{(P, Q\theta) \in \mathbb{S}t^* \oplus \mathbb{S}t^*\theta \mid Q \equiv 0 \pmod{\prod_{i=1}^n \alpha_i}\}$.

Example 4.5.3. The lens space $L_m(1, l_1, \dots, l_n)$, where $m > 1, l_1, \dots, l_n$ are positive integers with the greatest common divisor 1, is defined as the quotient of a $\mathbb{Z}/m\mathbb{Z}$ action on S^{2n+1} : $e^{2\pi i/m} \cdot (z_0, z_1, \dots, z_n) = (z_0, e^{2\pi i l_1/m} z_1, \dots, e^{2\pi i l_n/m} z_n)$. Since the T^n action on S^{2n+1} in the previous example commutes with the $\mathbb{Z}/m\mathbb{Z}$ action, the lens space $L_m(1, l_1, \dots, l_n)$ inherits an induced effective T^n action, as a quotient of S^{2n+1} by $\mathbb{Z}/m\mathbb{Z}$. The fixed-point set is a single circle $\{(z_1, 0, \dots, 0) \mid |z_1| = 1\}$ with isotropy weights $[\alpha_1], \dots, [\alpha_n]$, hence the T^n action on $L_m(1, l_1, \dots, l_n)$ is GKM. Moreover, we still have the formality criterion $\sum \dim H^*(L_m(1, l_1, \dots, l_n)^{T^n}) = 2 = \sum \dim H^*(L_m(1, l_1, \dots, l_n))$. The equivariant cohomology is again $H_{T^n}^*(L_m(1, l_1, \dots, l_n)) = \{(P, Q\theta) \in \mathbb{S}t^* \oplus \mathbb{S}t^*\theta \mid Q \equiv 0 \pmod{\prod_{i=1}^n \alpha_i}\}$.

Remark 4.5.1. We can equip S^{2n+1} with the standard contact form $\Theta = x_0 dy_0 - y_0 dx_0 + x_1 dy_1 - y_1 dx_1 + \dots + x_n dy_n - y_n dx_n$ where $x_j + iy_j = z_j$ and the induced contact form on $L_m(1, l_1, \dots, l_n)$. Note the contact form is invariant under the T^n action used in previous two examples and one can define moment maps for each generating vector field $\frac{\partial}{\partial \theta_j}$ of the torus T^n as $\Theta(\frac{\partial}{\partial \theta_j}) = x_j^2 + y_j^2$. The hyperplane bundle $(\text{Ker } \Theta, \omega = d\Theta)$ is symplectic and hence is a complex vector bundle. This gives T^n -invariant stable almost complex structure on S^{2n+1} and $L_m(1, l_1, \dots, l_n)$, so that the weights $\alpha_1, \dots, \alpha_n$ are determined with signs.

Remark 4.5.2. Remember that we are using \mathbb{Q} -coefficient and the ordinary cohomology $H^*(L_m(1, l_1, \dots, l_n), \mathbb{Q}) \cong H^*(S^{2n+1}, \mathbb{Q})$. It's not surprising to get $H_{T^n}^*(L_m(1, l_1, \dots, l_n), \mathbb{Q}) \cong H_{T^n}^*(S^{2n+1}, \mathbb{Q})$ for equivariant cohomology, though the formality criterion Theorem 2.2.2 only works for coefficients in $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ not for \mathbb{Z} .

Example 4.5.4. Take a product of an even dimensional T^k -equivariantly formal, GKM manifold M^{2m} and an odd dimensional T^l -equivariantly formal, GKM manifold N^{2n+1} . The new $2(m+n)+1$ -dimensional manifold $M \times N$ under the product action of $T^k \times T^l$ is also equivariantly formal and GKM. We can construct a GKM graph for $M \times N$ out of the graphs of M and N . For example, let's try $N = S^1$, and M an even dimensional, stable almost complex manifold with a T^k -action. Then the GKM graph for T^k action on $M \times S^1$ is obtained by replacing the \bullet -vertices of the graph of M into \circ -vertices and inserting a \square -vertex at the center of each edge of the graph of M . Since the weights of the standard S^1 -action on S^2 at the two poles are 1 and -1 , the weights of the T -action on the S_α^2 at the two poles N, S are α and $-\alpha$, so are the weights of the T -action on the $S_\alpha^2 \times S^1$ at the two fixed circles $\{N\} \times S^1, \{S\} \times S^1$. This means that in the relation \ddagger , we get $Q_N - Q_S \equiv 0 \pmod{\alpha}$, i.e. $Q_N \equiv Q_S \pmod{\alpha}$. If we denote E_M as the set of edges of the GKM graph of M , then $H_{T^k}^*(M \times S^1) = \{(P, Q\theta) : V \rightarrow \mathbb{S}t^* \oplus \mathbb{S}t^*\theta \mid P_x \equiv P_y \text{ and } Q_x \equiv Q_y \pmod{\alpha} \text{ for each edge } \overline{xy} \text{ with weight } [\alpha] \text{ in } E_M\} = H_{T^k}^*(M) \otimes H^*(S^1)$.

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